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Very Extended E_8 and A_8 at low levels, Gravity and Supergravity

Peter West

Department of Mathematics
King's College, London, UK

Abstract
We define a level for a large class of Lorentzian Kac-Moody algebras. Using this we find the representation content of very extended A_{D-3} and E_8 (i.e. E_{11}) at low levels in terms of A_{D-1} and A_{10} representations respectively. The results are consistent with the conjectured very extended A_8 and E_{11} symmetries of gravity and maximal supergravity theories given respectively in hep-th/0104081 and hep-th/0107209. We explain how these results provided further evidence for these conjectures.

email: pwest@mth.kcl.ac.uk

1. Introduction

One of the most surprising discoveries in supergravity theories was the realisation that the maximal supergravity theory in four dimensions possessed a hidden E_7 symmetry [1]. Hidden symmetries were found in the other maximal supergravity theories, and the lower the dimension of the maximal supergravity theory the higher the rank of the hidden symmetry and the more interesting it was. An account of these, with their original references, can be found in [2]. Over the years there have also been some other realisations of possible symmetry algebras in supergravity and string theories. It has been shown [3,4] that eleven dimensional supergravity does possess an SO(1,2)×SO(16) symmetry, although the SO(1,10) tangent space symmetry is no longer apparent in this formulation. It has also been noticed that some of the objects associated with the exceptional groups that appear in the reductions appear naturally in the unreduced theory [5]. In string theory it was found that the closed bosonic string reduced on the torus associated with the unique self-dual twenty six dimensional Lorentzian lattice is invariant under the fake monster algebra [6]. It has also been suggested [7] that there is some evidence of Kac-Moody structures in threshold correction of the heterotic string reduced on a six dimensional torus. Finally, it has been found that the maximal supergravity theory in eleven dimensions, in a special limit, possess one dimensional solutions which correspond to a particle mechanics in which the motion is restricted to the Weyl chambers of E_{10} [8].

Despite these observations it has been widely thought that the exceptional groups found in the dimensional reductions of the maximal supergravity theories can not be symmetries of eleven dimensional supergravity and must arise as a consequence of the dimensional reduction procedure. This is perhaps understandable given that the hidden symmetries were associated with scalar fields and there are no scalar fields in eleven dimensional supergravity. However, it has been conjectured [2] that the eleven dimensional supergravity theory possess a hidden E_{11} symmetry. It had previously been found [9] that the whole of the bosonic sector of eleven dimensional supergravity including its gravitational sector was a non-linear realisation. This placed the fields of the theory on a equal footing and naturally incorporated symmetries even though there were no scalars. This construction contained substantial fragments of larger symmetries and suggested that these should be incorporated into a Kac-Moody, or more general symmetry group. Even though it was not shown that such a symmetry was realised one could determine that this symmetry should contain very extended E_8, i.e. E_{11} [2]. It was also shown that this construction could be generalised to the ten dimensional IIA [2] and IIB [10] supergravity theories and the corresponding Kac-Moody algebra was also E_{11} in each case. It was also proposed that gravity in D dimensions should also have a hidden Kac-Moody symmetry which was very extended A_{D-3} [11].

These ideas were taken up by authors of reference [12] who considered eleven dimensional supergravity as a non-linear realisation of the E_{10} subalgebra of E_{11}, but in the small tension limit which played a crucial role in the work of reference [8]. These authors also introduced a new concept of level in the context of E_{10} which allowed them to deduce the representation content of E_{10} in terms of representations of A_9 at low levels. They showed that the eleven dimensional supergravity equations was E_{10} invariant, up to level three, in the small tension limit and provided one adopted a particular map relating the fields at a given spatial point to quantities dependent only on time. In this limit the spatial dependence of the fields was very restricted.

In this paper we generalise the notion of level given in reference [12] for a large class of Lorentzian Kac-Moody algebras. In section three we calculate the representation content of E_{11} and very extended A_{D-3} at low levels in terms of A_{10} and A_{D-1} representations respectively. We then apply these results in sections four and five to the conjectures that gravity in D dimensions in invariant under very extended A_{D-3} and eleven dimensional supergravity is invariant under E_{11} respectively.

2. Levels in Lorentzian Kac-Moody Algebras

We consider the particular class of Kac-Moody algebras discussed in section three of reference [13], namely an algebra whose Dynkin diagram C possess at least one node such that deleting it leads to a finite dimensional semi-simple Lie algebra. We denote the Dynkin diagram of this remaining Lie algebra by C_R. For simplicity, we will consider the case when this remaining algebra is irreducible, but the generalisation is obvious. Adopting the notation of reference [13]; we denote the preferred simple root by α_c and the simple roots of C_R by $\alpha_i, i=1,\ldots,r-1$. these may be written as [13]

$$\alpha_c = -\nu + x \tag{2.1}$$

where $\nu = -\sum_i A_{ci}\lambda_i$, x is a vector in a space orthogonal to the roots of C_R and λ_i are the fundamental weight vectors of C_R. We will assume that the simple roots have length squared two and then $2-\nu^2 = x^2$.

A positive root α of the Kac-Moody algebra C can be written as

$$\alpha = l\alpha_c + \sum_i m_i\alpha_i = lx + l\sum_i A_{ci}\lambda_i + \sum_{jk} A^f_{jk}\lambda_k \tag{2.2}$$

where A^f_{jk} is the Cartan matrix of C_R. We define the level, denoted l, of the roots of C to be the number of times the root α_c occurs. This agrees with the notion of level introduced in reference [12] for the case of E_{10}. For a fixed l, the root space of C can be described by its representation content with respect to C_R. We now discuss restrictions that one can place on the possible representations that can occur at a given level generalising the considerations of reference [12] for E_{10}. Given an irreducible representation with highest weight Λ it is completely specified by its Dynkin indices which are defined by $p_j = (\Lambda, \alpha_j) \geq 0$. It follows from equation (2.2) that a representation of C_R, with Dynkin indices p_j, is contained in C at level l if there exist positive integers m_k such that

$$p_j = lA_{cj} + \sum_k m_k A^f_{kj} \tag{2.3}$$

Any Kac-Moody algebra with symmetric Cartan matrix has its roots bounded by $\alpha^2 \leq 2, 1, 0\ldots$ [14]. Assuming we are dealing with such a Kac-Moody algebra, using equations (2.1) and (2.2), and the observation that $(2-\nu^2)\det A_{C_R} = A_C$ we find that

$$\alpha^2 = l^2\frac{\det A_C}{\det A_{C_R}} + \sum_{ij} p_i (A^f_{ij})^{-1} p_j \leq 2, 1, 0, -1, \ldots \tag{2.4}$$

Since the first term is fixed for a given l and the second term is positive definite this places constraints on the possible values of p_i.

Acting with the raising or lowering operators of C_R on the adjoint representation of C takes one from one root of C to a new root of C. The original and the new roots of C are of the form of equation (2.2) with l and m positive or negative integers depending if the root is in the positive of negative root space, However, under the action of such operators the first term of this equation, i.e. lx, is unchanged, and so the level of the new root is the same. As the positive or negative roots have all their coefficients of a given sign we find that the action of these raising or lowering operators does not take one out of the positive or negative root space of C. As a result, we may conclude that if a given representation of C_R occurs in C then it must be contained entirely in the positive roots with a copy in the negative root space of C. By considering a negative root, which has the form of equation (2.2), but with a negative sign on all terms on the right hand-side, we conclude that C may contain the representation of C_R with Dynkin indices p_j if

$$\sum_j (A^f_{kj})^{-1} p_j = -l \sum_j A_{cj} (A^f_{jk})^{-1} - m_k \tag{2.5}$$

where $m_k = 0, 1, 2, \ldots$. Since the left-hand side and the first term on the right-hand side are both positive, one finds, for fixed l, constraints on the allowed representations of C_R that can occur.

A solution to the constraints of equations (2.4) and (2.5) implies that the corresponding representation of C_R can occur in the root space of C, but it does not imply that it actually does occur. The above the constraints find the vectors in the lattice spanned by the positive or negative roots of C that have length squared $2, 1, 0, -1, \ldots$ and contain highest weight vectors of representations of the reduced subalgebra C_R. This is not the same as starting from the simple roots, taking their multiple commutators and imposing the Serre relations. By definition the latter calculation leads to all the roots of the Kac-Moody algebra and no more . However, given the list of possible representation that satisfy the constraints of equations (2.4) and (2.5) we can consider the set of associated generators and construct the Kac-Moody algebra C up to the level investigated by insisting that it satisfies all the consequences of the Serre relations. One simple requirement is that they satisfy the Jacobi identities. In some of the examples considered below one finds that this step implies that some of the potential representations of the reduced algebra do not actually belonging to the Kac-Moody algebra. In particular, we will see that some vectors contained in the root lattice which do satisfy the constraints are actually highest weight vectors of the Kac-Moody algebra C and do not actually belong to the Kac-Moody algebra. Indeed, the criterion given above does not distinguish between the Kac-Moody algebra C and the algebra derived from the physical state vertex operators associated with the roots lattice. As the latter contains many more states than the Kac-Moody algebra one expects to find these additional states by using the criterion above.

Finding the root multiplicities of Kac-Moody algebras is an unsolved problem except for a few exceptional cases. One may hope that the concept of level given above may provide a new avenue of attack on this problem.

3. Representations of very extended E_8 and A_{D-3} at low levels

In this section we find the representation content of very extended E_8 and A_{D-3} at low levels in terms of representations of A_{10} and A_{D-1} respectively. Let us begin with very extended E_8, i.e. E_{11}. The Dynkin diagram for this algebra is found by connecting ten dots in a horizontal line by a single line and then placing another dot above the third node from the right and connecting it with a single line to that node. The preferred node, c is the node above the horizontal line and deleting it gives A_{10}. Clearly, at level zero we have the adjoint representation of A_{10}. The inverse Cartan matrix of A_{D-1} is given by

$$(A^f_{jk})^{-1} = \begin{cases} \frac{j(D-k)}{11}, & j \leq k \\ \frac{k(D-j)}{D}, & j \geq k \end{cases} \tag{3.1}$$

Using this fact for $D = 11$, the constraint of equation (2.5) becomes

$$\sum_{j \leq k} j(11-k)p_j + \sum_{j>k} k(11-j)p_j = -11 n_k + l \begin{cases} 3(11-k), & k \geq 3 \\ 8k, & k = 1,2 \end{cases}, \tag{3.2}$$

where $n_k = 0, 1, 2 \ldots$. Analysing this equation and equation (2.4) using the relations $\det A_{E_n} = 9 - n$ and $\det A_{A_n} = n + 1$, it is straightforward, if tedious, to verify that they allow the following representations of A_{10}

$$l = 1,\ p_3 = 1 \ \ ; l = 2,\ p_6 = 1 \ \ ; l = 3,\ p_1 = 1, p_8 = 1 \text{ and } p_9 = 1 \ \ ;$$

$$l = 4,\ p_{10} = 1, p_1 = 2 \text{ and } p_{10} = 1, p_2 = 1 \text{ and } p_9 = 1, p_3 = 1 \tag{3.3}$$

All other p_j's being zero.

The corresponding generators are given by

$$K^a{}_b, R^{a_1 a_2 a_3}, R^{a_1 a_2 \ldots a_6}, R^{a_1 a_2 \ldots a_8, b} \text{ and } R^{a_1 a_2 \ldots a_9}, \tag{3.4}$$

where $R^{[a_1 a_2 \ldots a_8, b]} = 0$, at levels zero, one, two and three repectively. These generators are precisely those introduced in the non-linear realisation of eleven dimensional supergravity based on E_{11} [2] with the exception of the last generator, the $p_9 = 1$ representation at level three, which is absent from the E_{11} algebra as a result of the Jacobi identities[2]. They obey the commutation relations [2]

$$[K^a{}_b, K^c{}_d] = \delta^c_b K^a{}_d - \delta^a_d K^c{}_b, \tag{3.5}$$

$$[K^a{}_b, R^{c_1 \ldots c_6}] = \delta^{c_1}_b R^{a c_2 \ldots c_6} + \ldots,\ \ [K^a{}_b, R^{c_1 \ldots c_3}] = \delta^{c_1}_b R^{a c_2 c_3} + \ldots, \tag{3.6}$$

$$[R^{c_1 \ldots c_3}, R^{c_4 \ldots c_6}] = 2 R^{c_1 \ldots c_6}, \quad [R^{a_1 \ldots a_6}, R^{b_1 \ldots b_3}] = 3 R^{a_1 \ldots a_6 [b_1 b_2, b_3]}, \tag{3.7}$$

$$[R^{a_1 \ldots a_8, b}, R^{b_1 \ldots b_3}] = 0,\ [R^{a_1 \ldots a_8, b}, R^{b_1 \ldots b_6}] = 0,\ [R^{a_1 \ldots a_8, b}, R^{c_1 \ldots c_8, d}] = 0 \tag{3.8}$$

$$[K^a{}_b, R^{c_1 \ldots c_8, d}] = (\delta^{c_1}_b R^{a c_2 \ldots c_8, d} + \cdots) + \delta^d_b R^{c_1 \ldots c_8, a}. \tag{3.9}$$

The full E_{11} algebra up to level three is then given by these relations plus those involving the negative roots

$$R_{a_1 a_2 a_3}, R_{a_1 a_2 \ldots a_6}, R_{a_1 a_2 \ldots a_8, b} \tag{3.10}$$

The existence of the generators up to level three can also be inferred by the considerations of U-duality groups in the toroidal reduction of eleven dimensional supergravity [19].

As explained in the previous section, it can happen that there exist vectors in the root lattice that are solutions of the constraints, but are actually highest weight vectors of the Kac-Moody algebra. The fundamental weights of the Kac-Moody algebra are given by [13]

$$l_i = \lambda_i + \nu.\lambda_i \frac{x}{x^2},\ l_c = \frac{x}{x^2} \tag{3.11}$$

The $p_9 = 1$ representation at level three of E_{11} corresponds to the vector $-3x + \lambda_9$. Using equation (3.11) we recognise this vector as just the fundamental weight $l_9 = -3x + \lambda_9$ of E_{11}. Constructing the root string of l_9 we find that it contains the vector

$$l_9 - \alpha_9 - \ldots - \alpha_3 - \alpha_c = -4x + \lambda_2 + \lambda_{10}, \tag{3.12}$$

where in the last equation we have re-expressed the vector in the root string in terms of x and fundamental weights of A_{10}. Clearly, the highest weight representation of E_{11} contains at level four the $p_2 = 1, p_{10} = 1$, all other p_j's zero, representation of A_{10}. Examining the solutions of the constraints at level four of equation (3.3) we recognise that this is one of the representations that occurs and so it must also be eliminated from the E_{11} algebra.

We now consider very extended A_8. Its Dynkin diagram is given by drawing 10 dots on a horizontal line and joining them together by a single line, we then draw a node above the horizontal line and join it with a single line to the node on the far right and the third node from the left. The preferred node, labeled as c, is the one above the horizontal. Deleting this node leaves the nodes in the horizontal line which correspond to the algebra A_{10}. At level zero one has the adjoint representation of A_{10}. For this case equation (2.5) becomes

$$\sum_{j\le k} j(11-k)p_j + \sum_{j>k} k(11-j)p_j = -11m_k + l \begin{cases} (11+2k), & k \le 8 \\ 9(11-k), & k = 9, 10 \end{cases}, \tag{3.13}$$

where $m_k = 0, 1, 2 \ldots$. Analysing this equation and equation (2.4) implies that at the first two levels one has the potential representations

$$l = 1,\ p_1 = 1, p_8 = 1 \text{ and } p_9 = 1 \quad ;$$

$$l = 2,\ p_1 = 1, p_6 = 1,\ \text{and } p_2 = 1, p_5 = 1 \text{ and } p_8 = 1, p_{10} = 1,\ \text{and } p_1 = 1, p_7 = 1, p_{10} = 1 \tag{3.14}$$

all other p_j's zero. Hence we have the generators

$$K^a{}_b, R^{a_1\ldots a_8,b} \tag{3.15}$$

where $R^{[a_1\ldots a_8,b]} = 0$, at levels zero and one. We have not included a generator $R^{a_1\ldots a_9}$ since we will explain below that this is not actually in the very extended A_8 algebra.

In fact, very extended A_8 is a subalgebra of E_{11} since A_8 is a subgroup of E_8. As a result, all the A_{10} representations that occur as representations of very extended A_8 must also be representations of E_{11}. A detailed examination of the constraints of equations (2.4) applied to A_{10} and E_{11} reveals that a A_{10} representation of very extended A_8 at level l will be a representation of E_{11} at level $3l$. Furthermore, equation (2.5) becomes equations (3.2) and (3.13) for the cases of E_{11} and A_{10} respectively and we find that if we take the level in the former equation to be $3l$ where l is the level of very extended A_8 then equation (3.13) becomes equation (3.2) provided we identify

$$n_k = \begin{cases} m_k + l(2k-1), \; k = 1,2 \\ m_k + l(8-k), \; k = 3,\ldots,8 \\ m_k, \; k = 9,10 \end{cases} \tag{3.16}$$

Since if m_k is a positive integer so is n_k, we find that solutions to equation (3.13) at level l are solutions of equation (3.2) at level $3l$ as required. Indeed, one sees that the representations of equation (3.14) at level one occur in equation (3.3) at level three and one can explicitly check that the representations at level two of very extended A_8 occur as solutions of the E_{11} equations at level six.

Finally, we consider the case of very extended A_{D-3} which generalises the previous case to arbitrary rank. Its Dynkin diagram is given by drawing $D-1$ dots on a horizontal line and joining them together by a single line, we then draw a node above the horizontal line and join it with a single line to the node on the far right and the third node from the left. The preferred node, labeled as c, is the one above the horizontal. Deleting this node leaves the nodes in the horizontal line which correspond to the algebra A_{D-1}. At level zero one has the adjoint representation of A_{D-1} with generators $K^a{}_b$. Analysing equations (2.4) and equation (2.5) implies that at level one has the potential representations

$$l = 1, \; p_1 = 1, p_{D-3} = 1 \text{ and } p_{D-2} = 1, \tag{3.17}$$

all other p_j's zero. Hence, we have the generators

$$K^a{}_b, R^{a_1\ldots a_{D-3},b} \tag{3.18}$$

where $R^{[a_1\ldots a_{D-3},b]} = 0$, at levels zero and one. We have not included a generator $R^{a_1\ldots a_{D-2}}$ since, as we will explain below, this is not actually in the very extended A_{D_3} algebra. To derive this result we first realise that equation (2.5) implies results for $k = 1$ and $k = D-1$ which, when added together, imply that

$$\sum_{j=1}^{D-1} p_j = 2l, 2l-1, \ldots \tag{3.19}$$

Hence, at level one one finds that at most two p_j's can be non-zero. Equation (2.5) for $k = D-1$ implies that at level one $p_{D-1} = 0$ and the $k = 1$ constraint implies that if $p_1 = 1$ then the only solution has $p_{D-3} = 1$ at level one. To look for the other possible solutions we may take $p_1 = p_{D-1} = 0$, then adding the $k = 2$ constraint to the $k = D-2$ constraint, one finds that

$$2\sum_{j=2}^{D-2} p_j = 3l, 3l-1, \ldots. \tag{3.20}$$

Hence, at level one at most one of the remaining p_j's can be non-zero. Examining the remaining constraints one finds that the only remaining solution is that listed in equation (3.17). One can show that equations similar to (3.19) and (3.20) can also be derived for the case of E_{11}

Just like the case of E_{11}, the solution $p_{D-2} = 1$ of equation (3.17) is in fact not a representation that occurs in the very extended A_{D-3} Kac-Moody algebra. The corresponding vector in the root lattice is $-x + \lambda_{D-2}$ and, following the same arguments as above, we find that this vector is just the highest weight vector of the representation of very extended A_{D-3} with fundamental highest weight $l_{D-2} = -x + \lambda_{D-2}$. One finds that at level two this representation contains the vector $-2x + \lambda_{D-4} + \lambda_1 + \lambda_{D-1}$ which is the $p_{D-4} = 1, p_{D-1} = 1, p_1 = 1$ representation. One can verify that at level two this is a possible solution and so this solution must also be excluded from the very extended A_{D-3} algebra.

The generators of very extended A_{D-3} given in equation (3.18) obey the commutation relations

$$[K^a{}_b, K^c{}_d] = \delta_b^c K^a{}_d - \delta_d^a K^c{}_b, \tag{3.21}$$

$$[K^a{}_b, R^{c_1\ldots c_{D-3},d}] = (\delta_b^{c_1} R^{ac_2\ldots c_{D-3},d} + \cdots) + \delta_b^d R^{c_1\ldots c_{D-3},a}. \tag{3.22}$$

The complete very extended A_{D-3} algebra up to level one is then given by the above relations plus those involving the negative root generators

$$R_{c_1\ldots c_{D-3},b}. \tag{3.23}$$

4. Dual Gravity and Very extended A_{D-3}

The bosonic sector of eleven dimensional supergravity, including its gravitational sector, was shown [9] to be a non-linear realisation. It was subsequently, realised that this formulation possessed the Borel subgroup of E_7 as a symmetry and it was conjectured that this theory could be formulated in such a way that it was invariant under E_{11} [2]. Demanding that the E_8 borel subgroup and the A_{10} subgroup be symmetries implied that one must reformulate the theory so that gravity was described by a dual formulation involving the fields $h_a{}^b$ and $h_{a_1 a_2\ldots a_{D-3},b}$ [2]. An action for a dual theory of gravity in D dimensions was given [2] which was equivalent to Einstein's theory, it contained the vierbein $e_\mu^a = (e^h)_\mu{}^a$ and a field $Y_{a_1\ldots a_{D-2},b}$. The corresponding equations of motion were given by

$$(-1)^{D-2}\frac{1}{(D-3)!}\epsilon^{\mu\nu\tau_1\ldots\tau_{D-2}} Y_{\tau_1\ldots\tau_{D-2},d} = 2e(\omega_d,{}^{\mu\nu} + e_d{}^\nu \omega_c,{}^{c\mu} - e_d{}^\mu \omega_c,{}^{c\nu}) \tag{4.1}$$

where $e = \det e^{\mu a}$, $\omega_\rho,{}_c{}^d$ is the usual expression for the spin connection in terms of the vierbein, and

$$\epsilon^{\mu\tau_1\ldots\tau_{D-1}}\partial_{\tau_1} Y_{\tau_2\ldots\tau_{D-1},d} = \text{terms bilinear in } Y \tag{4.2}$$

The field $Y_{\tau_1\ldots\tau_{D-2},d}$ has been scaled by a numerical factor compared to that of reference [2].

At the linearised level, equation (4.2) implies that $Y_{\tau_1 \ldots \tau_{D-2},d}$ can be solved in terms of $h_{\tau_2 \ldots \tau_{D-2}},b$. Substituting this into equation (4.1) and writing $\omega_{\mu,cd}$ in terms of its standard linearised expression in terms of $e_\mu{}^b$ one finds the equation

$$(-1)^{D-2}\frac{(D-2)}{(D-3)!}\epsilon^{\mu\nu\tau_1\ldots\tau_{D-2}}\partial_{[\tau_1}h_{\tau_2\ldots\tau_{D-2}],d} = -\partial_\mu(e_{d\nu}+e_{\nu d})+\partial_d(e_{\mu\nu}+e_{\nu\mu})+\partial_\nu(e_{d\mu}+e_{\mu d})$$

$$+2\eta_{\nu d}(\partial_\mu e_c{}^c - \partial_c e_\mu{}^c) - 2\eta_{\mu d}(\partial_\nu e_c{}^c - \partial_c e_\nu{}^c) \tag{4.3}$$

. Taking ∂_μ implies the linearised Einstein equation. Carrying out a local Lorentz transformation $\delta e_{\mu\nu} = \lambda_{\mu\nu}$ one finds it is a symmetry provided $\delta h_{\tau_2\ldots\tau_{D-2},d} = -\epsilon_{\tau_2\ldots\tau_{D-2}dm\rho\kappa}\lambda^{\rho\kappa}$.

Based on these, and other considerations, it was proposed [11] that even pure gravity in D dimensions could be described as a non-linear realisation based on very extended A_{D-3}. The results of section three provide encouraging signs for this conjecture. Since eleven dimensional supergravity contains gravity, any conjecture for the Kac-Moody symmetries of gravity and eleven dimensional supergravity theories would have to be consistent. As explained in section three, this is the case since E_{11} contains very extended A_8 as a sub-algebra. Furthermore, the very extended A_{D-3} given in equation (3.17) contains the generators $K^a{}_b$ and $R^{a_1\ldots a_{D-3},b}$ at levels zero and these imply that the non-linear realisation is built from the fields $h_a{}^b, \hat{h}_{a_1\ldots a_{D-3},b}$ plus fields which appear at higher levels. The level zero and one fields are almost exactly those used in the above dual formulation of gravity. The difference is that the very extended A_8 algebra requires that $R^{[a_1\ldots a_{D-3},b]} = 0$ with a corresponding contraint for the field $\hat{h}_{a_1\ldots a_{D-3},b}$

It is instructive to derive Einstein's equation from the field equations rather than the action, as was the case in reference [2]. Since some of the manipulations are best carried out in form language and some in components we given both versions. We first consider the tensor

$$Y_{\rho_1\ldots\rho_{D-2},a} = \epsilon_{ab_1\ldots b_{D-1}}e_{[\rho_1}{}^{b_1}\ldots e_{\rho_{D-3}}{}^{b_{D-3}}\omega_{\rho_{D-2}]}{}^{b_{D-2}b_{D-1}}, \tag{4.4}$$

whose corresponding vector valued $D-2$ form is given by

$$Y_a = \epsilon_{ab_1\ldots b_{D-1}}e^{b_1}\wedge\ldots\wedge e^{b_{D-3}}\wedge w^{b_{D-2}b_{D-3}} \tag{4.5}$$

where $e^a = dx^\mu e_\mu{}^a$ and $w^{bc} = dx^\mu w_\mu{}^{bc}$. The equation of motion is given by

$$\epsilon^{\mu\nu\rho_1\ldots\rho_{D-2}}(\partial_\nu Y_{\rho_1\ldots\rho_{D-2},a} - M_{\nu\rho_1\ldots\rho_{D-2},a}) = 0, \quad \text{or} \quad dY_a = M_a. \tag{4.6}$$

where the vector valued $D-1$ form is defined by

$$M_{\rho_1\ldots\rho_{D-1},a} = -\epsilon_{ab_1\ldots b_{D-1}}\{(D-3)w_{[\rho_1}{}^{b_1}{}_f e_{\rho_2}{}^f e_{\rho_2}{}^{b_2}\ldots e_{\rho_{D-2}}{}^{b_{D-3}}w_{\rho_{D-1}]}{}^{b_{D-2}b_{D-1}}$$

$$+(-1)^{(D-3)}e_{[\rho_1}{}^{b_1}\ldots e_{\rho_{D-3}}{}^{b_{D-3}}w_{\rho_{D-2}}{}^{b_{D-2}f}w_{\rho_{D-1}]f}{}^{b_{D-1}}\} \tag{4.7}$$

or

$$M_a = -\epsilon_{ab_1\ldots b_{D-1}}\{(D-3)w^{b_1}{}_f\wedge e^f\wedge e^{b_2}\wedge\ldots\wedge e^{b_{D-3}}\wedge w^{b_{D-2}b_{D-1}}$$

$$+(-1)^{(D-3)}e^{b_1}\wedge\ldots\wedge e^{b_{D-3}}\wedge w^{b_{D-2}f}\wedge w_f{}^{b_{D-1}}\} \tag{4.8}$$

Evaluating equation (4.6) we find that it becomes

$$\epsilon_{ab_1\ldots b_{D-1}}\epsilon^{\mu\nu\rho_1\ldots\rho_{D-2}}\{\frac{(D-3)}{2}T_{\nu\rho_1}{}^{b_1}e_{\rho_2}{}^{b_2}\ldots e_{\rho_{D-3}}{}^{b_{D-3}}w_{\rho_{D-2}}{}^{b_{D-2}b_{D-1}}\}$$

$$-2\det e(D-3)!(-1)^{D-2}(R_a{}^\mu - \frac{1}{2}e_a{}^\mu R) = 0 \tag{4.9}$$

In these equations the torsion and Riemann tensors are given by

$$T^a = T_{\mu\nu}{}^a dx^\mu \wedge dx^\nu = 2(de^a + w^a{}_b \wedge e^b),$$

$$R^a{}_b = R_{\mu\nu}{}^a{}_b dx^\mu \wedge dx^\nu = 2(dw^a{}_b + w^a{}_c \wedge w^c{}_b),\ \ R_{\mu\nu}{}^\mu{}_b = R_{\nu b} \tag{4.10}$$

The first term in equation (4.9) vanishes as the spin connection has the usual expression in terms of the vierbein and so the torsion tensor $T_{\mu\nu}{}^a$ vanishes. Hence, we are left with the familiar equation for general relativity without matter. The above formulation of general relativity agrees with that given in reference [18] for four dimensional space-time where a connection with twistor theory was made that may prove useful in future.

The equation of motion of equation (4.6) is second order in space-time derivatives, but the above equations are written in a such way that they allows us to express them as a system of equations that is first order in space-time derivatives by introducing the fields

$$e_\mu^a = (e^h)_\mu{}^a, h_{a_1\ldots a_{D-3},b} \text{ and } k_{a_1\ldots a_{D-2},b} \tag{4.11}$$

The equations of motion are now given by

$$\epsilon^{\mu\nu\rho_1\ldots\rho_{D-2}}(Y_{\rho_1\ldots\rho_{D-2}a} - k_{\rho_1\ldots\rho_{D-2},a}) = \epsilon^{\mu\nu\lambda\rho_1\ldots\rho_{D-2}}\hat{D}_\lambda h_{\rho_1\ldots\rho_{D-3},a}$$

or

$$Y_a - k_a = dh_a + \Omega_a{}^b \wedge h_b \tag{4.12}$$

and

$$\epsilon^{\tau\lambda\rho_1\ldots\rho_{D-2}}\{\hat{D}_\lambda k_{\rho_1\ldots\rho_{D-2},a} - \Omega_{\lambda a}{}^c Y_{\rho_1\ldots\rho_{D-2},c} - M_{\lambda\rho_1\ldots\rho_{D-2},a}\} = 0,$$

$$\text{or } dk_a + \Omega_a{}^b k_b - \Omega_a{}^b Y_b - M_a = 0 \tag{4.13}$$

where

$$\hat{D}_\lambda h_{\rho_1\ldots\rho_{D-3},a} = \partial_\lambda h_{\rho_1\ldots\rho_{D-3},a} + \Omega_{\lambda a}{}^b h_{\rho_1\ldots\rho_{D-3},b}, \tag{4.14}$$

and

$$\Omega_{\lambda a}{}^b = (e^{-1}\partial_\lambda e)_a{}^b,\ h_a = h_{\rho_1\ldots\rho_{D-3},a}dx^{\rho_1}\wedge\ldots\wedge dx^{\rho_{D-3}},\text{ and } k_a = k_{\rho_1\ldots\rho_{D-2},a}dx^{\rho_1}\wedge\ldots\wedge dx^{\rho_{D-2}}. \tag{4.15}$$

Differentiating equation (4.12) with respect to ∂_ν and using equation (4.13) we find the usual equations of motion of gravity. The calculation is most easily carried out using the forms and the relation

$$d\Omega_a{}^b + \Omega_a{}^c \wedge \Omega_c{}^b = 0 \tag{4.16}$$

Hence, equations (4.12) and (4.13) imply Einstein's equations of general relativity and constitute an interacting dual theory of gravity in terms of equations which are first order in time derivatives. In fact, one can also use equations (4.12) and (4.13) with the $\Omega_a{}^b$ terms set to zero and find the same result.

Let us now consider a non-linear realisation of very extended A_{D-3}, but including only fields of level zero and one. We therefore have the generators

$$K^a{}_b, R^{a_1\ldots a_8,b}, \tag{4.17}$$

We will take the local subgroup to be the Chevalley invariant subgroup and so the group element takes the form

$$g = e^{x^\mu P_\mu} e^{h_a{}^b K^a{}_b} e^{\hat{h}_{a_1\ldots a_{D-3},b} R^{a_1\ldots a_{D-3},b}} \tag{4.18}$$

where $\hat{h}_{[a_1\ldots a_{D-3},b]} = 0$. The Cartan forms are given by

$$\mathcal{V} = g^{-1}dg - \omega = dx^\mu (e_\mu{}^a P_a + \Omega_{\mu a}{}^b K^a{}_b + \tilde{D}_\mu \hat{h}_{a_1\ldots a_{D-3},b} R^{a_1\ldots a_{D-3},b}) - \omega \tag{4.19}$$

where $\omega = \frac{1}{2} dx^\mu \omega_{\mu bc} J^{bc}$ and

$$\tilde{D}_\mu \hat{h}_{a_1\ldots a_{D-3},b} = \partial_\mu \hat{h}_{a_1\ldots a_{D-3},b} + \Omega_{\mu a_1}{}^c \hat{h}_{c\ldots a_{D-3},b} + \ldots + \Omega_{\mu b}{}^c \hat{h}_{a_1\ldots a_{D-3},c} \tag{4.20}$$

We treat the Lorentz part of the subgroup in a preferred manner by including in g all of $K^a{}_b$ and as a result introducing the above spin connection. As discussed in reference [9], one can solve for the spin connection in terms of the Cartan form $\Omega_{\mu a}{}^b$ which a way that is uniquely determined by conformal invariance.

Examining equations (4.12) and (4.13) when written in terms of tangent indices we find that they are indeed formulated in terms of the above Cartan forms, namely $e_\mu{}^a$, $\Omega_{ca}{}^b$, or $\omega_{ca}{}^b$, and

$$\tilde{D}_{a_1} \hat{h}_{a_2\ldots a_{D-2},b} = e_{a_1}{}^{\mu_1} \ldots e_{a_{D-2}}{}^{\mu_{D-2}} (\partial_{\mu_1} \hat{h}_{\mu_2\ldots\mu_{D-2},b} + \Omega_b{}^c \hat{h}_{\mu_2\ldots\mu_{D-2},b}),$$

plus the fields $k_{a_1\ldots a_{D-2},b}$ and $h_{[a_1\ldots a_{D-3},b]}$. Clearly, describing gravity by a non-linear realisation based on the group very extended A_{D-3} with its subgroup taken to be the Cartan involution invariant subgroup implies, modulo a miracle involving the inverse Higgs mechanism, that one would have an infinite number of fields or Goldstone bosons. Since we do not possess such a system at present we can at best hope to find a system that has the fields of levels zero and one and is invariant under the action of the generators of very extended A_{D-3} up to level one. In the above, we have found a system of equations that is equivalent to Einstein's general relativity, involves the required fields and is invariant under the Borel subgroup of very extended A_{D-3} up to level one. It is clearly invariant under the Lorentz group, but it remains to show that it is invariant under the remaining generator at level one. The fields $k_{a_1\ldots a_{D-2},b}$ and $h_{[a_1\ldots a_{D-3},b]}$ play the role of the infinite number of fields occurring at levels two and above that we must add in order to have the full non-linear realisation.

Of course, given the Cartan forms of equation (4.19) one would not immediately conclude that the required equations of motion were those of equations (4.12) and (4.13). However, as explained in reference [9] one must demand simultaneous invariance under the conformal group. Carrying out this requirement should result in equations (4.12) and (4.13) being the unique equations invariant under both algebras up to the level being considered. This, and its extension to the next level of very extended A_8, would make the calculation given in this section more compelling.

We close this section by generalising the above formulation of gravity to include matter and in particular consider a rank p gauge field $A_{(p)} = A_{\mu_1\ldots\mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}$ with field strength $F_{(p+1)} = (p+1)dA_{(p)}$. The dual field strength being given by $G_{(D-p-1)} = \star F_{(p+1)}$. To incorporate matter we define

$$\hat{M}_a = M_a + \frac{1}{(D-p-1)} G_{(D-p-1)} \wedge F_{(p)a} - (-1)^p \frac{1}{(p+1)} G_{(D-p-2)a} \wedge F_{(p+1)}) \qquad (4.21)$$

where $F_{(p)a} = F_{\mu_1\ldots\mu_p a} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}$ and similarly for $G_{(D-p-2)a}$. If we replace M_a by $\hat{M}_a$ in equation (4.6) or equation (4.13) we find that

$$R_\mu{}^d - \frac{1}{2} e_\mu{}^d R = c\{F_{\mu\lambda_1\ldots\lambda_p} F^{d\lambda_1\ldots\lambda_p} - \frac{1}{2(p+1)} e_\mu{}^d F_{\lambda_1\ldots\lambda_{p+1}} F^{\lambda_1\ldots\lambda_{p+1}}\} \qquad (4.22)$$

where c is a constant which can be adjusted by rescaling $A_{(p)}$. We recognise the right hand side as the required contribution to the energy-momentum tensor. As must be the case for consistency of the equations of motion, and as is required by the conservation of energy and momentum, one can verify that $d\hat{M}_a = 0$.

We observed that the above formulation of gravity involves the field $h_{a_1\ldots a_{D-3},b}$ which is not subject to a constraint $h_{[a_1\ldots a_{D-3},b]} = 0$. If we suppose that the equation to describe the dual formulation of gravity must be first order in space-time derivatives; then, at the linearised level, it must be of the form

$$\epsilon^{\mu\nu\tau_1\ldots\tau_{D-2}} \partial_{\tau_1} h_{\tau_2\ldots\tau_{D-2},}{}^d + \ldots = f \partial^\mu h^{\nu d} + \ldots \qquad (4.23)$$

where f is a constant and $+\ldots$ stands for all possible terms linear in the space-time derivatives and in the same field, but with different index structure. Furthermore, one must be able to apply a derivative to the left hand-side of the above equation and be able to eliminate the field $h_{a_1\ldots a_{D-3},b}$ and obtain an equation in terms of $h_a{}^b$ alone which is the linearised Einstein equation. Writing down the most general terms and imposing the constraint $h_{[a_1\ldots a_{D-3},b]} = 0$ we find constraints on the coefficients that are not compatible with deriving the linearised Einstein equation from equation (4.3). However, equations (4.1) and (4.2), at the linearised level, do provide a system of equations which lead to the linearised Einstein equation, but they involve the unconstrained field $h_{a_1\ldots a_{D-3},b}$.

It is instructive to discuss the dual formulation of gravity given above with those given in references [15-17]. In these references a dual formulation of gravity is given at the linearised level by an equation which is second order in space-time derivatives of the form

$$R_{\mu\nu}{}^{ab} = \epsilon_{\mu\nu}{}^{\rho_1\ldots\rho_{D-2}} (\partial_a \partial_{[\rho_1} l_{\rho_2\ldots\rho_{D-2}],b} - (a \to b)), \qquad (4.24)$$

where $l_{[a_1 \ldots a_{D-3},b]} = 0$. Using equation (4.1) one can calculate $R_{\mu\nu}{}^{ab}$ in terms of $Y_{\tau_1 \ldots \tau_{D-2},d}$ and, after a field redefinition, it can be brought to the form of equation (4.24). This can be seen to be a consequence of the fact that an additional space-time derivative allows an additional gauge symmetry which can be used to algebraically gauge it away, or equivalently, eliminate its appearance in the appropriate variables. This gauge symmetry is related to the "local Lorentz" transformation discussed below equation (4.3). Hence, from this view point the field $h_{[a_1 \ldots a_{D-3},b]}$ is not really present. However, to place gravity on the same footing as the gauge fields we would like to have a dual formulation of gravity that is expressed by equations that are first order in space-time derivatives. The gauge field $A_{a_1 \ldots a_3}$ requires only one additional field $A_{a_1 \ldots a_6}$ to rewrite its equations in terms of equations involving only one space-time derivative. However, the gravitational equations are essentially non-linear equations and this might be viewed as the source of the infinite number of fields required by the conjectured very extended A_8 symmetry. It would be interesting to see how the fields at the next level of very extended A_8 symmetry replace $h_{[a_1 \ldots a_{D-3},b]}$ and the other additional field $k_{[a_1 \ldots a_{D-2},b]}$ even at the linearised level.

5 Eleven Dimensional Supergravity and E_{11}

In this section we sketch how, starting from E_{11}, we can find eleven dimensional supergravity as a non-linear realisation. This contrasts with reference [2] where we started with the non-linear realisation of reference [9], identified E_{11}, and then found substantial fragments of an E_{11} symmetry. Most of the equations given below can be found in reference [2], but it is encouraging to see them emerge starting from E_{11}. In section three we found that the first three levels of E_{11} contain the generators in equations (3.5-3.9) plus the generators associated with negative roots given in equation (3.10). In a non-linear realisation all generators in the symmetry algebra not in the local subgroup must correspond to fields in the theory. The only exception to this is when one can use the so called inverse Higgs effect, but this mechanism will not concern us here. As explained in references [9,2], the level one and two generators $R^{a_1 a_2 a_3}$ and $R^{a_1 \cdots a_6}$ correspond to the gauge fields $A_{a_1 a_2 a_3}$ and its dual $A_{a_1 \ldots a_6}$ respectively. Gravity can be described by the the group $GL(11)$ and so corresponds to the level zero generators $K^a{}_b$ associated with the fields $h^a{}_b$. However, as realised in reference [2], demanding that E_8 and A_{10} be symmetry groups implies that we should use a dual formulation of gravity which involves the field $\hat{h}_{a_1 \ldots a_8,b}$ corresponding to the generator $R^{a_1 \cdots a_8,b}$. Thus, there is a very close correspondence between the fields introduced to describe eleven dimensional supergravity as a non-linear realisation and the fields content up to level three required by demanding E_{11} as a symmetry group. This can be taken as yet another encouraging sign that the maximal supergravity theories really are invariant under E_{11}.

Taking the Cartan involution invariant subgroup as the local subgroup, up to level three, the non-linear realisation is constructed from the group element [9,2]

$$g = e^{h_a{}^b K^a{}_b} e^{\frac{1}{3!} A_{a_1 \ldots a_3} R^{a_1 \cdots a_3}} e^{\frac{1}{6!} A_{a_1 \ldots a_6} R^{a_1 \cdots a_6}} e^{\hat{h}_{a_1 \ldots a_{D-3},b} R^{a_1 \cdots a_{D-3},b}} \tag{5.1}$$

The equations of motion should be constructed from the Cartan forms $\mathcal{V} = g^{-1} dg - w$ and they are given by [9,2]

$$\mathcal{V} = dx^\mu ({e_\mu}^a P_a + {\Omega_{\mu a}}^b {K^a}_b + \frac{1}{3!}\tilde{D}_\mu A_{c_1\ldots c_3} R^{c_1\ldots c_3} + \frac{1}{6!}\tilde{D}_\mu A_{c_1\ldots c_6} R^{c_1\ldots c_6} + \tilde{D}_\mu \hat{h}_{c_1\ldots c_8,b} R^{c_1\ldots c_8,b}) - \omega \tag{5.2}$$

where

$$\begin{gathered} {e_\mu}^a \equiv {(e^h)_\mu}^a, \ \tilde{D}_\mu A_{c_1\ldots c_3} \equiv \partial_\mu A_{c_1c_2c_3} + ({(e^{-1}\partial_\mu e)_{c_1}}^b A_{bc_2c_3} + \ldots), \\ \tilde{D}_\mu A_{c_1\ldots c_6} \equiv \partial_\mu A_{c_1\ldots c_6} + ({(e^{-1}\partial_\mu e)_{c_1}}^b A_{bc_2\ldots c_6} + \ldots) - 20(A_{[c_1\ldots c_3}\tilde{D}_\mu A_{c_4\ldots c_6]}) \\ \tilde{D}_\mu \hat{h}_{c_1\ldots c_8,b} = \partial_\mu \hat{h}_{c_1\ldots c_8,b} + ({(e^{-1}\partial_\mu e)_{c_1}}^d \hat{h}_{dc_2\ldots c_8,b} + \ldots) \\ - \frac{1}{(3!)^3} A_{[c_1\ldots c_3}\tilde{D}_\mu A_{c_4c_5c_6}A_{c_7c_8]b} - \frac{1}{(3!)(6!)} A_{[c_1\ldots c_6}\tilde{D}_\mu A_{c_7c_8]b} - tr \end{gathered} \tag{5.3}$$

where $+\ldots$ denotes the action of $(e^{-1}\partial_\mu e)$ on the other indices and $-tr$ means that one should subtract a term in order to ensure that $\tilde{D}_\mu h_{[c_1\ldots c_8,b]} = 0$.

As explained in references [2,9] we require invariance not only under E_{11} but also under the conformal group. This implies that we should use only those combinations of E_{11} Cartan forms which are covariant under the latter group. The result of this procedure for the rank three and six forms is that one should only use the simultaneously covariant forms [2,9]

$$\tilde{F}_{c_1\ldots c_4} \equiv 4({e_{[c_1}}^\mu \partial_\mu A_{c_2\ldots c_4]} + {e_{[c_1}}^\mu {(e^{-1}\partial_\mu e)_{c_2}}^{\ \ b} A_{bc_3c_4]} + \ldots) \tag{5.4}$$

and

$$\tilde{F}_{c_1\ldots c_7} \equiv 7({e_{[c_1}}^\mu (\partial_\mu A_{c_2\ldots c_7]}) + {e_{[c_1}}^\mu {(e^{-1}\partial_\mu e)_{c_2}}^{\ \ b} A_{bc_3\ldots c_7]} + \ldots + 5\tilde{F}_{[c_1\ldots c_4}\tilde{F}_{c_5\ldots c_7]}) \tag{5.5}$$

The invariant equation of motion is then given by

$$\tilde{F}^{c_1\ldots c_4} = \frac{1}{7!}\epsilon_{c_1\ldots c_{11}}\tilde{F}^{c_5\ldots c_{11}} \tag{5.6}$$

There only remains the equation for gravity. This equation should be given by equation (4.12) but with $\hat{D}_\mu h_{c_1\ldots c_8,b}$ of equation (4.14) replaced by $\tilde{D}_\mu \hat{h}_{c_1\ldots c_8,b}$ of equation (5.3) and the addition of the field $h_{[c_1\ldots c_8,b]}$. Taking d of this equation one does indeed find terms similar to those on the right-hand side of equation (4.21) which are those required to find the energy momentum tensor of the four form field strength on the right-hand side of the Einstein equation. However, one also finds expressions which are not written in terms of the four form field strength or its dual and these should be canceled by terms added to the definitions of Y_a and M_a. We hope to return to this calculation in the future.

The level in E_{11} is measured by the generator $\frac{1}{3}D = \frac{1}{3}\sum_a {K^a}_a$ where ${K^a}_b$ are the generators of GL(11). Clearly, D commutes with the Cartan subalgebra generators and simple roots of SL(11), but with the simple root generator $E_c = R^{91011}$ one has the commutator $[D, E_c] = E_c$ as required. However, the generator D corresponds to a symmetry of eleven dimensional supergravity acting with e^{rD} on the group element of equation (5.1) one finds that x^μ scales and ${h_a}^b \to {h_a}^b + r{h_a}^b$ with the other fields being inert. It is then easy to see that the Cartan forms referred to the tangent space, such as in equation (5.4) and (5.5), are inert under D transformations. As such, the D symmetry is automatically preserved by the equations of motion constructed from the Cartan forms. In fact, in a Lorentz covariant formulation the level of the fields does not appear to provided a very useful means ordering the calculation. This is apparent from equation (2.14) which relates fields of different levels, proceeding level by level implies that at the first level the right-hand side of equation (5.7) vanishes.

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